The General Linear Model

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Some Simple Problems

2*x=10
What's x?

2*x + 5=10
What's x?

2*x + 5 = y
y=10
What are x and y?

x+y=1
x-y=1
What are x and y?

x=1
x=2
x=6

What's x?

x+y = 1

What are x and y?

We Can Do This in Matlab

2*x=10
What's x?

2*x + 5=10
What's x?

2*x + 5 = y
y=10
What are x and y?

x+y=1
x-y=1
What are x and y?

[2]*x=[10]

[2]*x =[5]
What's x?

[2 -1; 0 1]*[x y] = [-5 10]

[1 1; 1 -1]*[x y] = [1 1]

=> "inv", "PINV"

x=1
x=2
x=6

What's x?

x+y = 1

What are x and y?

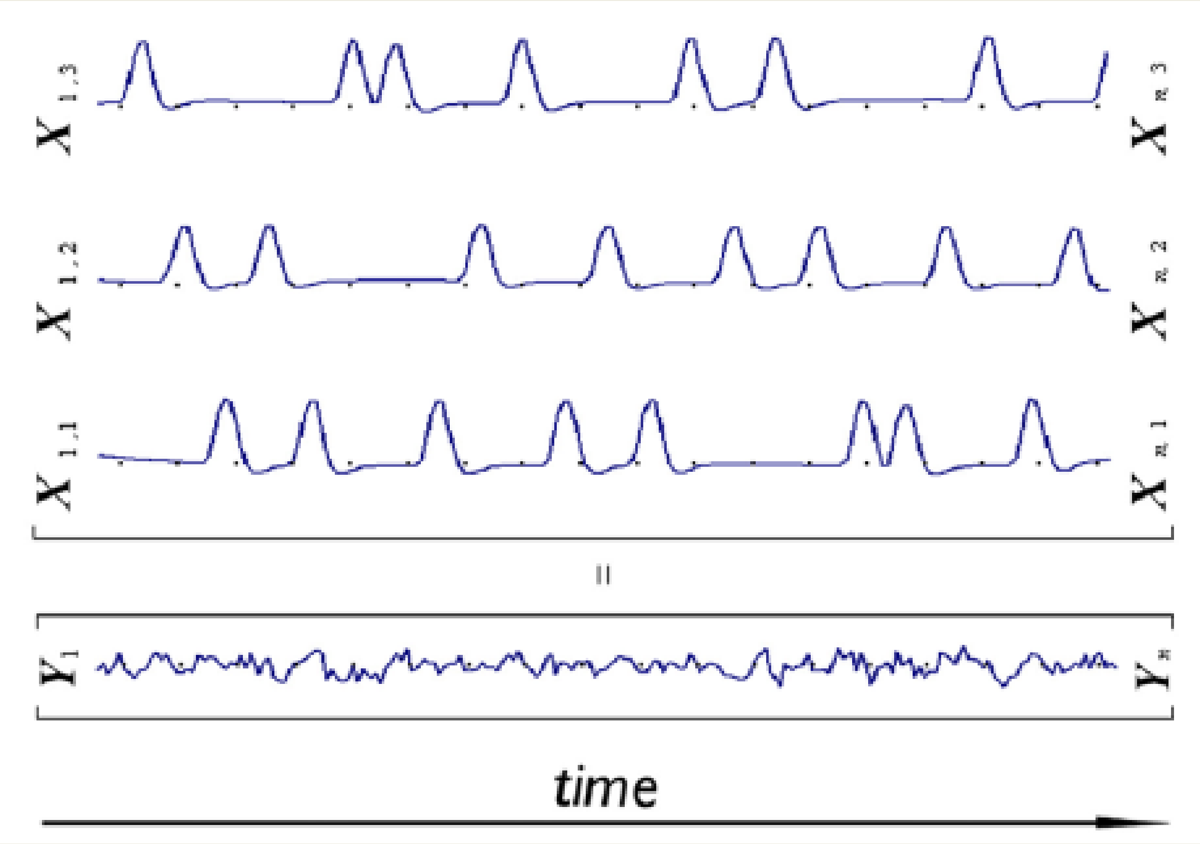
fMRI General Linear Model

Predicted time course for event type 1

Predicted time course for event type 2

Predicted time course for event type 3

BOLD time course in one voxel



measured time series

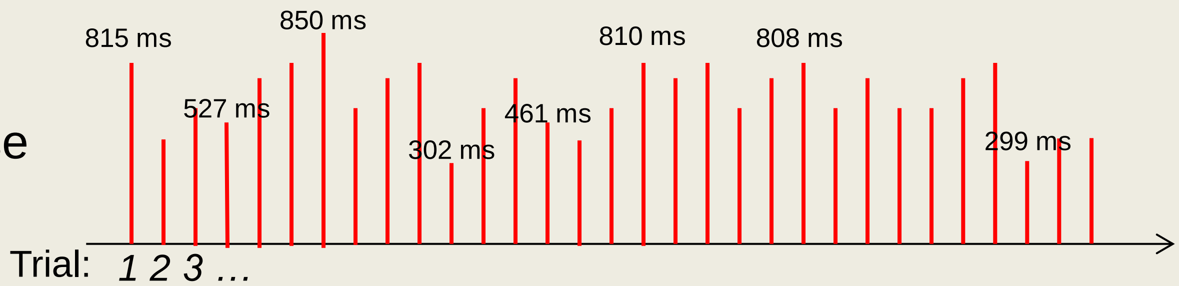
parameter estimates

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta}$$

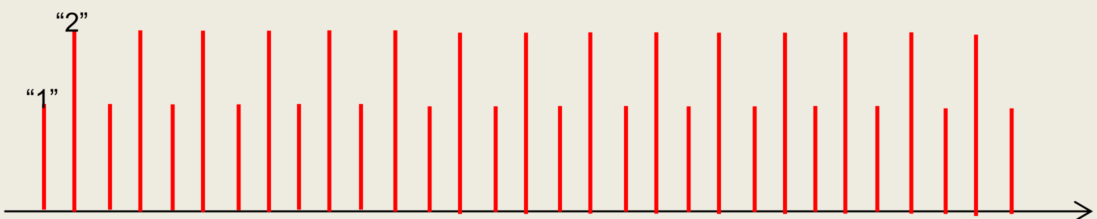
design matrix

Regression of Reaction Times

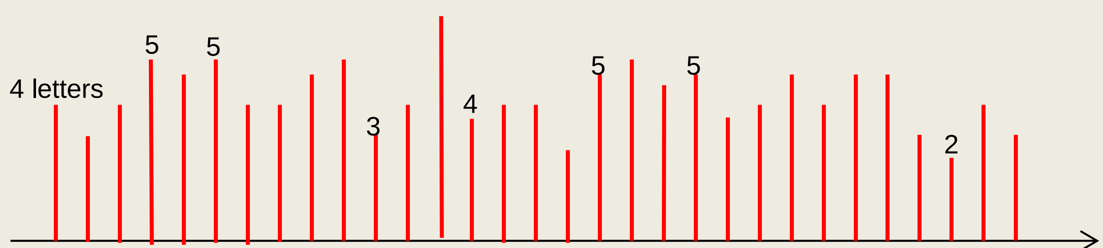
Which variables can explain these data?



Repetition?



Word length?



RTs

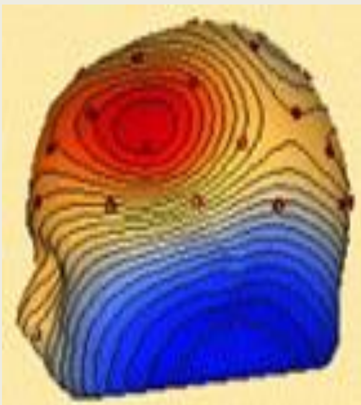
$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta}$$

regression coefficients

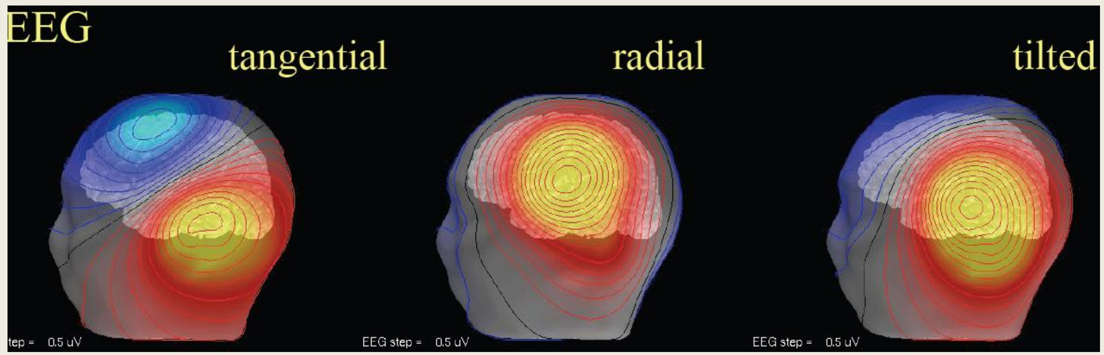
predictor variables

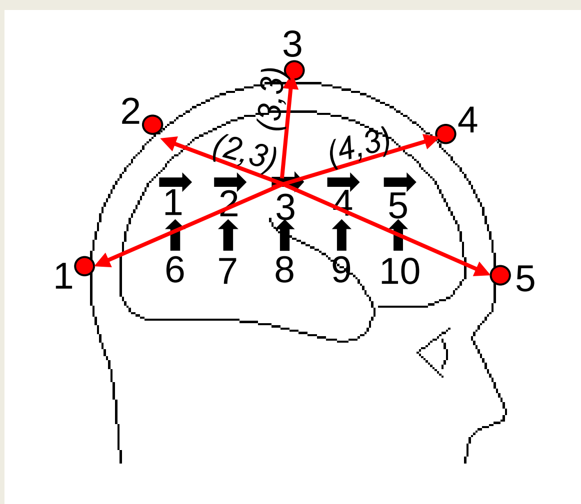
EEG/MEG "Inverse Problem"

Which sources explain this topography?



Maybe these?





data vector

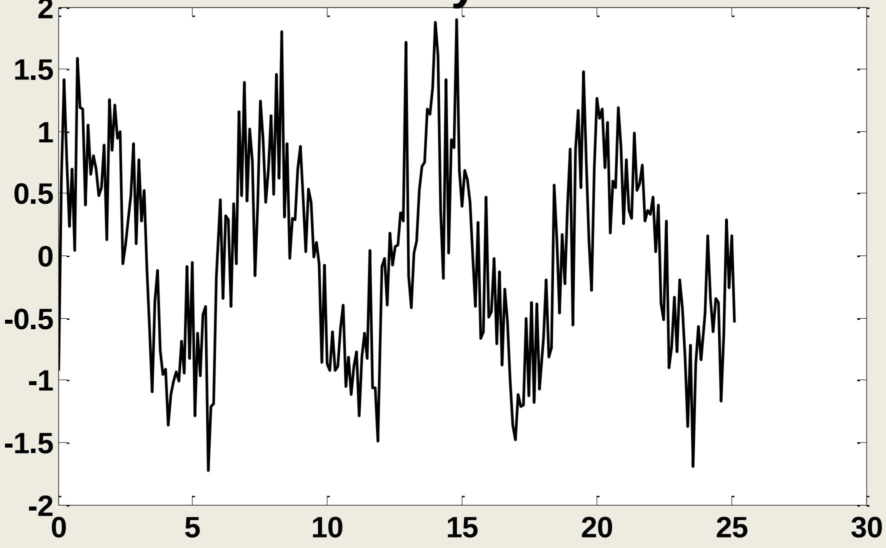
$$\mathbf{d} = \mathbf{L}\mathbf{s}$$

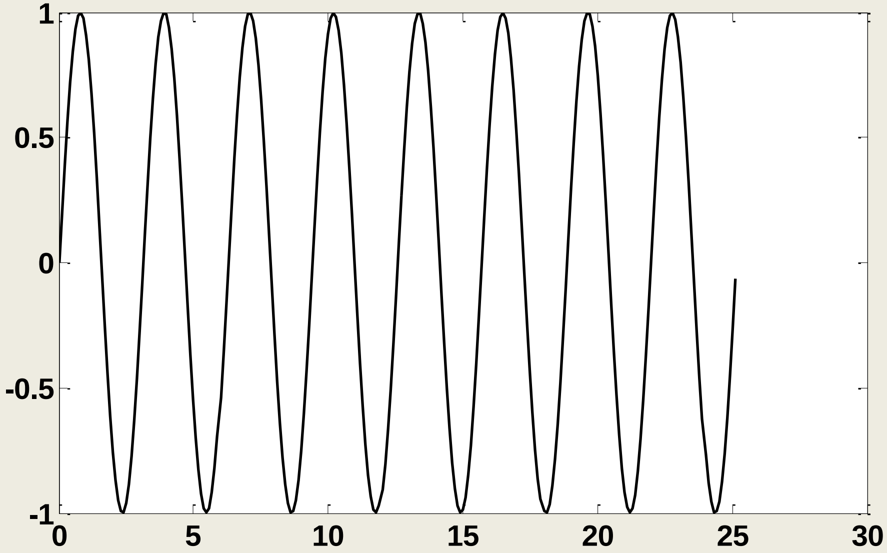
source vector

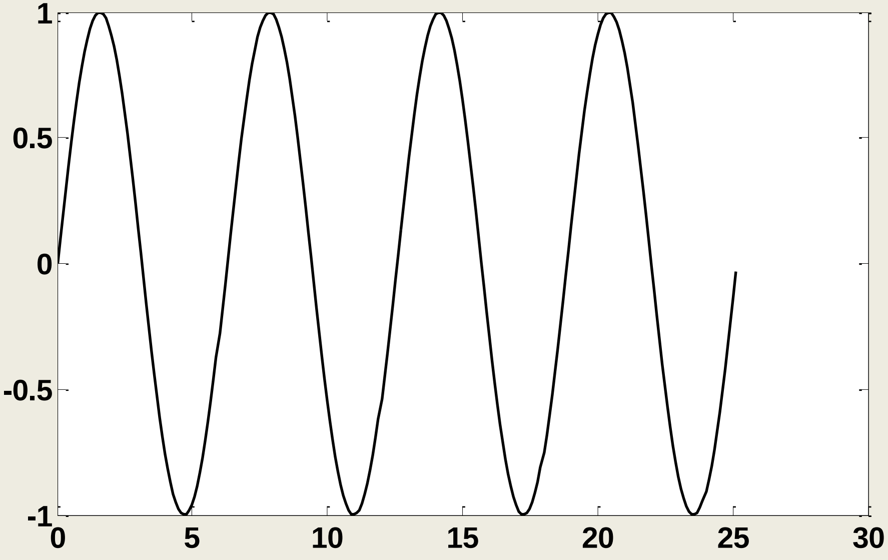
Leadfield matrix

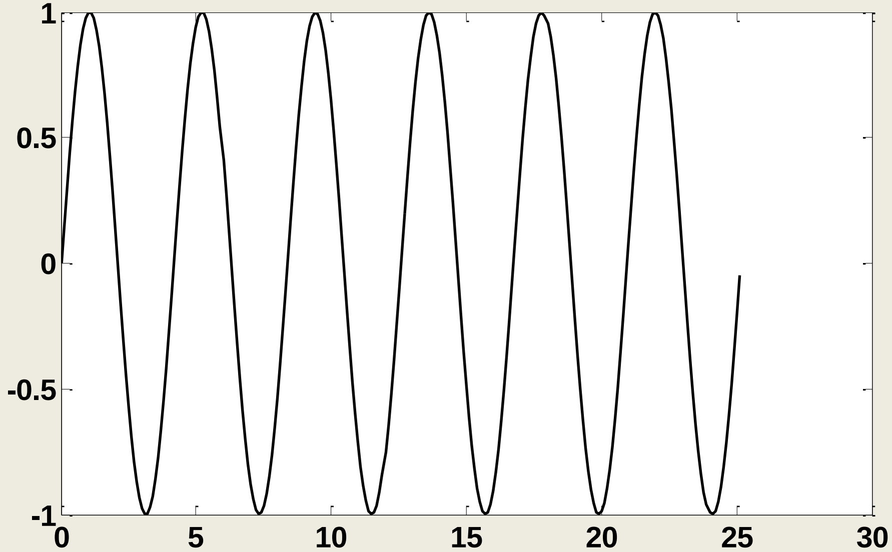
Choosing the right basis functions

These are your data:



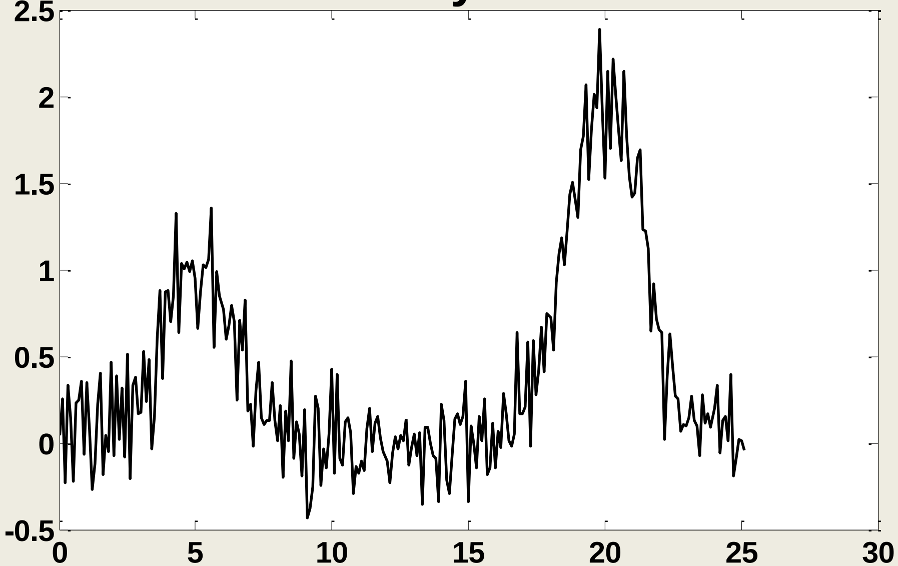


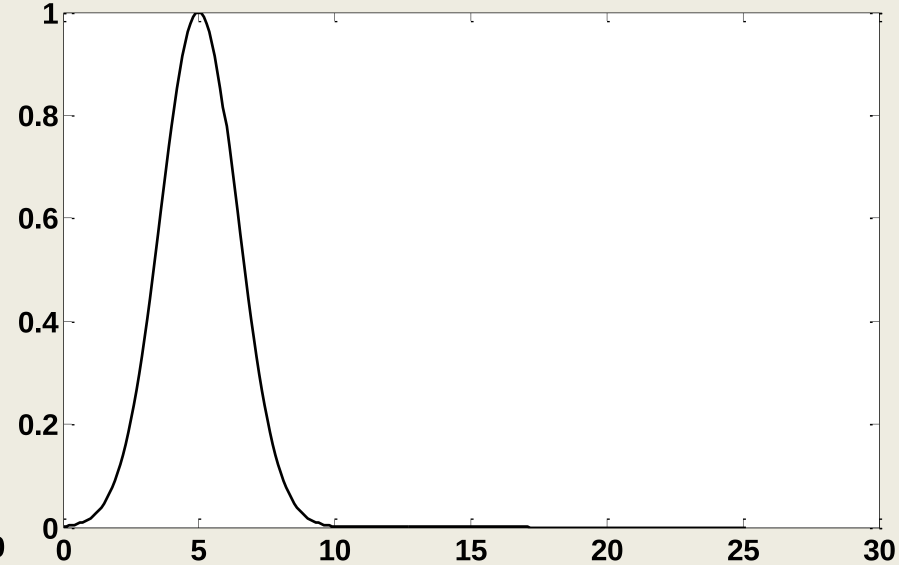


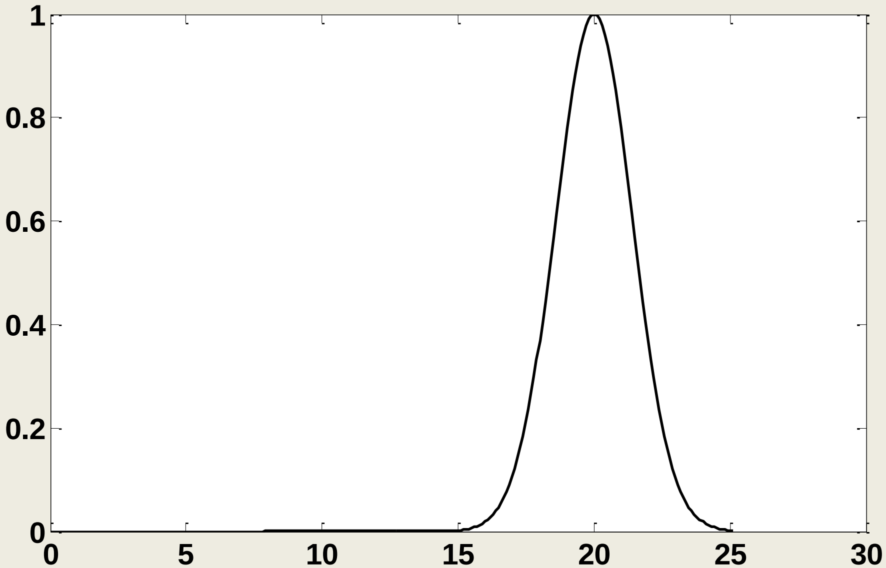


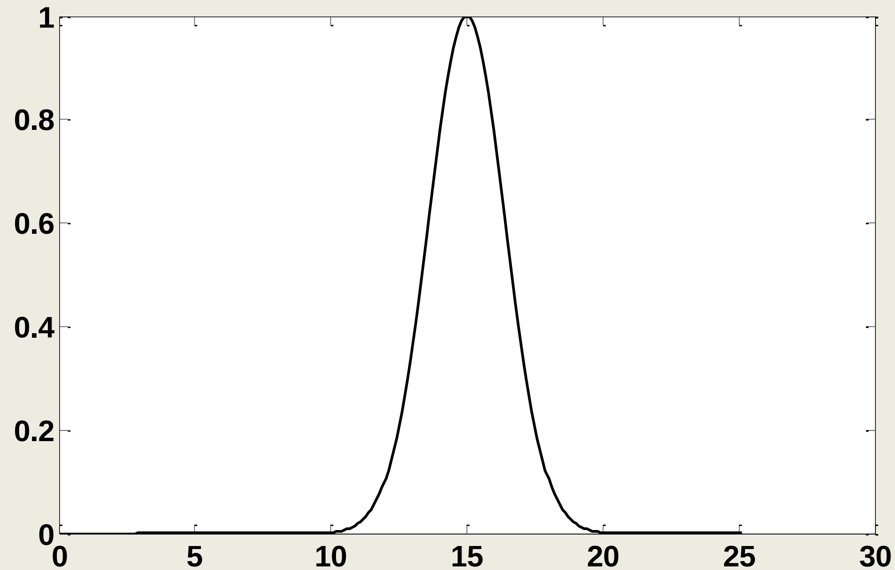
Choosing the right basis functions

These are your data:









The choice of the right basis functions depends on the problem and what you know about it

– it’s not about the math

Basis Functions

Every vector with *n* elements can be decomposed into *n* different independent vectors

There are many such decompositions – some may be more useful than others

[1 2 3]
can be decomposed as

1*[1 0 0] + 2*[0 1 0] + 3*[0 0 1]

or

1*[1 1 1] + 1*[0 1 0] + 2*[0 0 1]

or

-1*[1 -1 0] + -1*[0 1 -1] + 2*[1 1 1]

The vectors, or "basis functions", are usually given,
and we are looking for the coefficients that explain the data

The "Inverse Problem" – Parameter Estimation

Usually, we have
1) The data
2) A set of basis functions

=> We want to know:
How much do the different basis functions contribute to our measured data?

If possible, we want to describe the relationship between our desired parameters and the data in the framework of the General Linear Model:

$$\mathbf{d} = \mathbf{X}\mathbf{b}$$

For example:

1) Data: [1 2 3]
2) Basis functions: [1 -1 0] [0 1 -1] [1 1 1]
3) Problem: What are *a,b,c* for [1 2 3] = *a**[1 -1 0] + *b**[0 1 -1] + *c**[1 1 1]

The Basic Idea Behind Linear Parameter Estimation

1) Data: [1 2 3]

2) Basis functions: [1 -1 0] [0 1 -1] [1 1 1]

3) Problem: What are *a,b,c* for [1 2 3] = *a**[1 -1 0] + *b**[0 1 -1] + *c**[1 1 1]

This may mean:

- basis functions ([1 -1 0] etc.): your predicted fMRI time courses for different conditions
- a/b/c: the “betas” for different conditions
- y the measured fMRI time course per voxel

- basis functions: predictor variables, one value per stimulus (length, frequency…)
- a/b/c: regression coefficients for different conditions
- y: measured reaction times for all stimuli

- basis functions: EEG/MEG topographies for point sources (dipoles)
- a/b/c: source strengths for those point sources
- y: measured topography at a particular latency, “component” etc.

Remember?

$$\mathbf{y} = \begin{pmatrix} y_1 \\ \dots \\ y_j \\ \dots \\ y_C \end{pmatrix} = \mathbf{Mx} = \begin{pmatrix} M_{11} & \dots & M_{1j} & \dots & M_{1C} \\ \dots & \dots & \dots & \dots & \dots \\ M_{i1} & \dots & M_{ij} & \dots & M_{iC} \\ \dots & \dots & \dots & \dots & \dots \\ M_{R1} & \dots & M_{Rj} & \dots & M_{RC} \end{pmatrix} \begin{pmatrix} x_1 \\ \dots \\ x_j \\ \dots \\ x_C \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{C} x_j * M_{1j} \\ \dots \\ \sum_{j=1}^{C} x_j * M_{ij} \\ \dots \\ \sum_{j=1}^{C} x_j * M_{Rj} \end{pmatrix} = \sum_{j=1}^{C} x_j * \begin{pmatrix} M_{1j} \\ \dots \\ M_{ij} \\ \dots \\ M_{Rj} \end{pmatrix} = \sum_{j=1}^{C} x_j \mathbf{M}_{.j}$$

$\mathbf{M}_{.j}$ stands for the j-th column of $\mathbf{M}$.

Each column of **M** is weighted by the corresponding element in **x**.

y is a "linear combination" of the columns of **M.**

$$a * \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + b * \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + c * \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} * \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

You could call **M** the "design matrix".

Solving Linear Equations

Problem:

- We have an equation **Mx=y**
- We know M and y
- We want to know x

If only we had a matrix $\mathbf{M}^{-1}$ with the property

$$\mathbf{M}^{-1}*\mathbf{M} = \mathbf{I}$$

(**I** is the identity matrix)

because then:

$$\mathbf{M}^{-1}*\mathbf{M}\mathbf{x} = \mathbf{I}*\mathbf{x} = \mathbf{x} = \mathbf{M}^{-1}\mathbf{y}$$

$\mathbf{M}^{-1}$ is the “inverse matrix” **of M**

(Not every matrix has a unique inverse matrix. If it does, it’s called “invertible”)

Example

Linear Equations

$$\mathbf{y} = \mathbf{x} + 2 \;\Rightarrow\; \mathbf{x} = ? \qquad \begin{pmatrix} y_1 \\ \dots \\ y_j \\ \dots \\ y_C \end{pmatrix} = \mathbf{y} = \mathbf{x} + 2 = \begin{pmatrix} x_1 + 2 \\ \dots \\ x_j + 2 \\ \dots \\ x_C + 2 \end{pmatrix} \;\Rightarrow\; \mathbf{x} = \mathbf{y} - 2 = \begin{pmatrix} x_1 - 2 \\ \dots \\ x_j - 2 \\ \dots \\ x_C - 2 \end{pmatrix}$$

$$\mathbf{y} = 2 * \mathbf{x} \;\Rightarrow\; \mathbf{x} = ? \qquad \begin{pmatrix} y_1 \\ \dots \\ y_j \\ \dots \\ y_C \end{pmatrix} = \mathbf{y} = 2 * \mathbf{x} = \begin{pmatrix} 2 * x_1 \\ \dots \\ 2 * x_j \\ \dots \\ 2 * x_C \end{pmatrix} \;\Rightarrow\; \mathbf{x} = \mathbf{y} / 2 = \begin{pmatrix} y_1 / 2 \\ \dots \\ y_j / 2 \\ \dots \\ y_C / 2 \end{pmatrix}$$

Linear Equations

Problem:

$$\begin{aligned} x_1 + x_2 &= 1 \qquad & x_1 &= ? \\ x_1 - x_2 &= 1 \qquad & x_2 &= ? \end{aligned}$$

"Basis functions"

Matrix notation:

$$\begin{pmatrix} 1 * x_1 + 1 * x_2 \\ 1 * x_1 - 1 * x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{M} * \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Solution?

("2*x=1 => x = ½")

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}} = \frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\mathbf{M}} = \mathbf{M}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

What is the "inverse" of a matrix?

Inverse Matrices

Definition: A matrix multiplied by its inverse is the identity matrix:

$$\mathbf{I} = \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{matrix}$$

$$\mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$$

(just like (1/3)*3 = 1)

$$\mathbf{M} * \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

2*x=1

=> ½*2*x= ½

=> x = ½

$$\Downarrow$$

$$\underbrace{\mathbf{M}^{-1}\mathbf{M}}_{identity} * \mathbf{x} = \mathbf{M}^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \mathbf{x} = \mathbf{M}^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

“Orthonormal” basis functions

Orthonormal: Orthogonal and of unit norm/length

For example:
[1 0] and [0 1] are orthonormal basis functions

$$\alpha\begin{bmatrix}1\\0\end{bmatrix}+\beta\begin{bmatrix}0\\1\end{bmatrix}=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$$

$$\Downarrow$$

$$\alpha=\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=1$$

$$\beta=\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=1$$

No “inversion” necessary – just multiply basis functions to your data.

But: Often basis functions are not orthonormal

Problem of multiple linear regression:

If basis functions are correlated, the whole system of equations

needs to be taken into account

$\Rightarrow$ Matrix inversion is necessary

(“partialling out” variables)

“Linearly independent”: vectors are not perfectly correlated

“Orthogonal”: correction of vectors is exactly zero

“Overdetermined Problem” (e.g. Regression)

$$
\begin{aligned}
1*x_1+1*x_2&=1\\
2*x_1+1*x_2&=-1\\
2*x_1+2*x_2&=2\\
3*x_1+1*x_2&=0\\
3*x_1+2*x_2&=1.5\\
3*x_1+3*x_2&=2.5
\end{aligned}
\qquad
\begin{aligned}
x_1&=?\\
x_2&=?
\end{aligned}
$$

$$
\begin{pmatrix}1\\2\\2\\3\\3\\3\end{pmatrix}*x_1+\begin{pmatrix}1\\1\\2\\1\\2\\3\end{pmatrix}*x_2=\underbrace{\begin{pmatrix}1&1\\2&1\\2&2\\3&1\\3&2\\3&3\end{pmatrix}}_{\mathbf{M}}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\mathbf{Mx}=\underbrace{\begin{pmatrix}1\\-1\\2\\0\\1.5\\2.5\end{pmatrix}}_{\mathbf{d}}=\mathbf{d}
$$

M is not invertible, there is no unique solution for **x**.

We can find the **x** that minimises the least-squares error: $\|\mathbf{Mx}-\mathbf{d}\|^2=\min$

The matrix that provides this least-squares solution is the “pseudoinverse” of **M:** $\mathbf{M}^-$
(in Matlab: “pinv”)

$$
\mathbf{M}^- = \begin{matrix} -0.0526 & 0.1579 & -0.1053 & 0.3684 & 0.1053 & -0.1579\\ 0.1158 & -0.1474 & 0.2316 & -0.4105 & -0.0316 & 0.3474 \end{matrix}
$$

"Overdetermined Problem" (e.g. Regression)

If $\mathbf{Mx} = \mathbf{d}$ is an overdetermined problem (more data than unknowns), then

$$\mathbf{M}^{-} = (\mathbf{M}^{T}\mathbf{M})^{-1}\mathbf{M}^{T}$$

is the unique "(minimum-norm) least squares" solution.

$\mathbf{M}^{-}$ is also called the "Pseudoinverse of $\mathbf{M}$, in Matlab lingo "pinv".

“Underdetermined Problem” (e.g. EEG/MEG Inverse Problem)

$$1*x_1 + 1*x_2 + 1*x_3 = 1$$
$$1*x_1 + 2*x_2 + 3*x_3 = -1$$

$$x_1 = ?$$
$$x_2 = ?$$
$$x_3 = ?$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} * x_1 + \begin{pmatrix} 1 \\ 2 \end{pmatrix} * x_2 + \begin{pmatrix} 1 \\ 3 \end{pmatrix} * x_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{Mx} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \mathbf{d}$$

M is not invertible, there is no unique solution for **x**.
We can find the **x** that minimises the “norm” of the solution:

$$\mathbf{Mx} = \mathbf{d}$$
$$\|\mathbf{x}\|^2 = x_1^2 + x_2^2 + x_3^2$$

The matrix that provides this least-squares solution is the “pseudoinverse” of **M:** $\mathbf{M}^-$
(in Matlab: “pinv”)

$$\mathbf{M}^- = \begin{matrix} 1.3333 & -0.5000 \\ 0.3333 & 0.0000 \\ -0.6667 & 0.5000 \end{matrix}$$